How-To – Solving Systems of Linear Equations with Matrices

Systems of linear equations of the form: $\begin{matrix} ax + by = e \\ cx + dy = f \end{matrix}$ can be written as an equation involving matrices.

$\begin{matrix} ax + by = e \\ cx + dy = f \end{matrix}$ can be written in a matrix form: $\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$

Using multiplication: $\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$ This final equation looks like $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$

We write the equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$ in the form AX = B, where

A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the **coefficient** matrix, X = $\begin{bmatrix} x \\ y \end{bmatrix}$ is the **variable** matrix, and B = $\begin{bmatrix} e \\ f \end{bmatrix}$ is the **constant** matrix.

To find the solution of the system, (x, y), we need to isolate the variable matrix.

We do this by multiplying each side by a matrix by A^{-1}, the inverse of matrix A.

So $AX = B$ becomes $A^{-1} \cdot AX = A^{-1} \cdot B$.

It is true that any matrix multiplied by its inverse results in the identity matrix or $A^{-1} \cdot A = I$.

It is also true that any matrix multiplied by the identity matrix results in the original matrix.

So $A^{-1} \cdot AX = I \cdot X = X$. The final equation becomes $X = A^{-1} \cdot B$.

The matrix X is the solution to the problem.

EX: Solve the following system of linear equations, $\begin{matrix} 2x + y = 7 \\ 4x - 3y = -1 \end{matrix}$

The system in matrix form is: $\begin{bmatrix} 2 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

Step 1: Find the inverse of the coefficient matrix.

$A = \begin{bmatrix} 2 & 1 \\ 4 & -3 \end{bmatrix}$ $|A| = \begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} = 2 \times (-3) - 4 \times 1 = -6 - 4 = -10$

So $A^{-1} = \frac{1}{-10} \begin{bmatrix} -3 & -1 \\ -4 & 2 \end{bmatrix}$

Step 2: Mulitiply A^{-1} and B.

$$A^{-1}B = \frac{1}{-10} \begin{bmatrix} -3 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \end{bmatrix} = \frac{1}{-10} \begin{bmatrix} -3 \times 7 + -1 \times -1 \\ -4 \times 7 + 2 \times -1 \end{bmatrix} = \frac{1}{-10} \begin{bmatrix} -21 + 1 \\ -28 - 2 \end{bmatrix} = \frac{1}{-10} \begin{bmatrix} -20 \\ -30 \end{bmatrix} = \begin{bmatrix} -20/-10 \\ -30/-10 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So the solution is (2, 3).